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A Simpler Proof of the Characterization of Quadric CMC Hypersurfaces in S^{n+1}

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Abstract

In this short article, we present a new and simpler proof of a characterization of the quadric constant mean curvature hypersurfaces of the Euclidean sphere S^{n+1}, originally due to Alias, Brasil and Perdomo

Keywords

Euclidean sphere; Constant mean curvature hypersurfaces; Support functions; Totally umbilical hypersurfaces; Clifford torus

Introduction

In 2008, Alias, Brasil and Perdomo studied complete hypersurfaces immersed in the unit Euclidean sphere $S^{n+1} \subset \mathbb{R}^{n+2}$, whose height and angle functions with respect to a fixed nonzero vector of the Euclidean space $\mathbb{R}^{n+2}$ are linearly related. Let us recall that, for a fixed arbitrary vector $a \in \mathbb{R}^{n+2}$ the height and the angle functions naturally attached to a hypersurface $\psi : \Sigma^n \to S^{n+1}$ endowed with an orientation ν are defined, respectively, by $l_a = \langle \psi, a \rangle$ and $f_a = \langle \nu, a \rangle$. In this setting, they showed the following characterization result concerning the quadric constant mean curvature hypersurfaces of S^{n+1} [1,2]:

Theorem 1

Let $\psi : \Sigma^n \to S^{n+1} \subset \mathbb{R}^{n+2}$ be a complete hypersurface immersed in S^{n+1} with constant mean curvature. $l_a = \lambda f_a$ for some non-zero vector $a \in \mathbb{R}^{n+2}$ and some real number λ, then Σ^n is either a totally umbilical hypersurface or a Clifford torus $S^k(\rho) \times S^{n-k}\left(\sqrt{1-\rho^2}\right)$, for some k = 0; 1;..; n and some k=0,1,…,n and ρ>0.

Later on, working with a different approach of that used in [2], the first and second authors characterized the totally umbilical and the hyperbolic cylinders of the hyperbolic space H^{n+1}as the only complete hypersurfaces with constant mean curvature and whose support functions with respect to a fixed nonzero vector a of the Lorentz-Minkowski space are linearly related (see Theorem 4:1 of [3,4], for the case that a is either space like or time like, and Theorem 4:2 of [5], for the case that a is a nonzero null vector). In this short article, our purpose is just to use a similar approach of that in [4,5] in order to present a new and more simple proof of Theorem 1 (cf. Section 3). For this, in Section 2 we recall some preliminaries facts concerning hypersurfaces immersed in S^{n+1}.

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Preliminaries

Let $\psi : \Sigma^n \to S^{n+1} \subset \mathbb{R}^{n+2}$ be an orientable hypersurface immersed in the Euclidean sphere. We will denote by A the Weingarten operator of Σ^n with respect to a globally defined unit normal vector ν.

In order to set up the notation, let us represent by ∇^0, $\overline{\nabla}$ and ∇ the Levi-Civita connections of $\mathbb{R}^{n+2}$, S^{n+1} and Σ^n respectively. Then the Gauss and Weingarten formulas for Σ^n in S^{n+1} are given, respectively, by

$$\nabla^0{}_X Y = \nabla_X Y + \langle AX, Y \rangle \nu - \langle X, Y \rangle \psi$$

and

$$AX = -\overline{\nabla}_X \nu = -\nabla^0{}_X \nu,$$

for all tangent vector fields X, Y $\mathfrak{X}(\Sigma)$.

In what follows, we will work with the first three symmetric elementary functions of the principal curvatures $\lambda_1, \ldots \lambda_n$ of ψ, namely:

$$S_1 = \sum_i \lambda_i,\ S_2 = \sum_{i<j} \lambda_i \lambda_j \text{ and } S_3 = \sum_{i<j<k} \lambda_i \lambda_j \lambda_k$$

where $i, j, k \in \{1, \ldots, n\}$.

As before, for a fixed arbitrary vector $a \in \mathbb{R}^{n+2}$ let us consider the height and the angle functions naturally attached to which are defined, respectively, by $l_a = \langle \psi, a \rangle$ and $f_a = \langle \nu, a \rangle$. A direct computation allows us to conclude that the gradient of such functions are given by $\nabla l_a = a^T$ and $\nabla f_a = -A(a^T)$, where a^T is the orthogonal projection of a onto the tangent bundle $T\Sigma$, that is,

$$a^T = a - f_a \nu - l_a \psi$$

Taking into account that $\nabla^0 a = 0$ and using Gauss and Weingarten formulas, we obtain $\nabla_X A(a^T) = f_a AX + l_a X$ for all $X \in \mathfrak{X}(M)$. We use this previous identity jointly with Codazzi equation to deduce that

$$\nabla_X A(a^T) = f_a A^2 X + l_a AX + (\nabla_{a^T} A)(X),$$

For all that $X \in \mathfrak{X}(M)$. Thus according to [6] (see also [3]), it follows from the last two identities that

$$\nabla l_a = nHf_a - nl_a \tag{2.1}$$

$$\nabla f_a = -|A|^2 f_a + nHl_a - n\langle \nabla H, a^T \rangle, \tag{2.2}$$

where $H = (1/n) S_1$ is the mean curvature function of Σ^n

For what follows, it is convenient to consider the so-called Newton transformation

$$P_1 : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$$

$$P_1 = S_1 - A$$

where I is the identity operator. Naturally associated with the Newton transformation P_1, we have the Cheng-Yau's square operator [7], which is the second order linear di erential operator $\square : D(\Sigma) \to D(\Sigma)$ given by

$$\square \to h = tr(P_1 \circ \nabla^2 h) \tag{2.4}$$

Here $\nabla^2 h : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ stands for the self-adjoint linear operator metrically equivalent to the hessian of h, and it is given by

$$\langle \nabla^2 h(X), Y\rangle \langle \nabla_X(\nabla h), y\rangle$$

For all X, Y∈ $\mathfrak{X}(\Sigma)$.

Based on Reilly's seminal paper [8-10], Rosenberg [6] showed the following idenfitities related to the action of $\square$ on the functions l_a and f_a:

$$\square l_a = 2s_2 f_a - (n-1)s_1 l_a \qquad (2.5)$$

And

$$\square f_a = -(s_1 s_2 - 3s_3) f_a + 2s_2 l_a - \langle \nabla s_2, a^{\mathrm{T}}\rangle \qquad (2.6)$$

To close this section, we quote a suitable Simons-type formula which can be found in [1] or [11].

$$\square s_1 = \Delta s_2 + |\nabla s_1|^2 + 2s_2(|A| - n) - s_1^2 s_2 + 3s_1 s_3 + (n-1)s_1^2 \qquad (2.7)$$

Proof of Theorem

Now, we are in position to proceed with our alternative proof of Theorem 1.1. If λ=0 then $l_a = \lambda f_a = 0$, that is

$$\left\langle \psi(x), \frac{a}{|a|}\right\rangle = \frac{1}{|a|}\langle \psi(x), a\rangle = 0$$

for all $x \in \Sigma^n$ and, consequently, Σ^n is a totally umbilical sphere of S^{n+1}.

So, let us assume that λ≠0. We have $\Delta l_a = \lambda \Delta f_a$ and using the fact that H is constant, from (2.1) and (2.2) we conclude that

$$nHf_a - nl_a = -\lambda |A|^2 f_a + \lambda nHl_a$$

Or equivalently,

$$S_1 f_a - nl_a = -\lambda(S_1^2 - 2S_2) f_a + \lambda S_1 l_a = -\lambda S_1^2 f_a + 2\lambda S_2 f_a + \lambda S_1 l_a$$

Hence, we get that

$$S_1 f_a - nl_a + \lambda S_1^2 f_a - 2\lambda S_2 f_a - \lambda S_1 l_a = 0 \qquad (3.1)$$

By (3.1), we obtain

$$\begin{aligned} 0 &= \lambda(S_1 f_a - nl_a + \lambda S_1^2 f_a - 2\lambda S_2 f_a - \lambda S_1 l_a) \\ &= S_1(\lambda f_a) - n\lambda l_a + \lambda S_1^2(\lambda f_a) - 2\lambda S_2(\lambda f_a) - \lambda^2 S_1 l_a \\ &= S_1 l_a - n\lambda l_a + \lambda S_1^2 l_a - 2\lambda S_2 l_a \\ &= (S_1 - n\lambda + \lambda S_1^2 - 2\lambda S_2 - \lambda^2 S_1) l_a \end{aligned}$$

Thus,

$$(S_1 - n\lambda + \lambda S_1^2 - 2\lambda S_2 - \lambda^2 S_1) l_a = 0 \qquad (3.2)$$

We define a function

$h : \Sigma^n \to \mathbb{R}$ by

$$h(p) : (S_1 - n\lambda + \lambda S_1^2 - 2\lambda S_2 - \lambda^2 S_1)(p)$$

Suppose that $h(p_0) \neq 0$ for some $p_0 \in \Sigma^n$ Since h is smooth, there exists a neighbourhood u of p_0 in Σ^n in which $h(p) \neq 0$ all $p \in u$ From (3.2) we conclude that $l_a = 0$ in u and, hence $f_a = 0$ in u. since λ≠0. we arrive at a contradiction because in Σ^n we have

$$|\nabla l_a|^2 + f_a + l_a^2 = |a|^2 > 0$$

Therefore, $h = 0$ on Σ^n, that is,

$$S_1 - n\lambda + \lambda S_1^2 - 2\lambda S_2 - \lambda^2 S_1 \equiv 0 \qquad (3.3)$$

Consequently, S_2 is constant on Σ^n. Repeating the previous argument for the operator L_1 and using the fact that S_2 is constant, we also obtain that

$$2S_2 - \lambda(n-1)S_1 + \lambda S_1 S_2 - 3\lambda S_3 - 2\lambda^2 S_2 \equiv 0 \qquad (3.4)$$

We observe that the above equation shows that S_3 is also constant on Σ^n. We also note that this argument shows, in fact, that S_r is a constant function on Σ^n for all 2≤r≤n. From (2.7) we get

$$|\nabla A|^2 + 2S_2(S_1^2 - 2S_2 - n) - S_1(S_1 S_2 - 3S_3 - (n-1)S_1 = 0$$

More precisely,

$$|\nabla A|^2 + S_1^2 S_2 - 4S_2^2 - 2nS_2 + 3S_1 S_3 + (n-1)S_1^2 = 0 \qquad (3.5)$$

We observe that if H=0, then S_1=0 and, consequently, $|A|^2 = -2s_2$ From (3.3), we have $2S_2 = -n$ and $|A|^2 = n$. Therefore, since

$$\frac{1}{2}\Delta |A|^2 = n|A|^2 - |A|^4 + |\nabla A|^2$$

We have that $|\nabla A|^2 = 0$ and, hence, from Theorem 4 of [9], we conclude that Σ^n must be a Clifford torus

$S^k(\rho) \times S^{n-k}(\sqrt{1-\rho^2})$, for some k=0,1,…,n and some ρ>0.

Now, suppose that H≠0 By equation (3.4) we get

$$S_1(2S_2 - \lambda(n-1)S_1 + \lambda S_1 S_1 - 3\lambda S_3 - 2\lambda^2 S_2) = 0 \qquad (3.6)$$

that is,

$$2S_1 S_2 - \lambda(n-1)S_1^2 + \lambda S_1^2 S_2 - 3\lambda S_1 S_3 - 2\lambda^2 S_1 S_2) = 0 \qquad (3.7)$$

From equation (3.5) we have

$$\lambda |\nabla A|^2 + \lambda S_1^2 S_2 - 4\lambda S_2^2 - 2n\lambda S_2 + 3\lambda S_1 S_3 + \lambda(n-1)S_1^2 = 0 \qquad (3.8)$$

Furthermore, from a straightforward computation we can verify that

$$\lambda |\nabla A|^2 + 2S_1 S_2 + 2\lambda S_1^2 S_2 - 2\lambda^2 S_1 S_2 - 4\lambda S_2^2 - 2n\lambda S_2 = 0 \qquad (3.9)$$

Hence, if $S_2 = 0$ we obtain of (3.9) that $\lambda |\nabla A|^2 = 0$ consequently, $|\nabla A|^2 = 0$ and, since Σ^n is complete, it follows once more from Theorem 4 of [9] that Σ^n must be a Clifford torus.

If $S_2 \neq 0$ then $2S_2.(S_1 - n\lambda + \lambda S_1^2 - 2\lambda S_2 - \lambda^2 S_1) = 0$ implies

$$2S_1 S_2 - 2n\lambda S_2 + 2\lambda S_1^2 S_2 - 4\lambda S_2^2 - 2\lambda^2 S_1 S_2) = 0 \qquad (3.10)$$

We note that (3.10) and (3.9) imply $\lambda |\nabla A|^2 = 0$ and, hence, repeating the previous argument we also get that Σ^n is a Clifford torus. Therefore, we conclude the proof of Theorem 1.

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